Math 246A Lecture 15 Notes

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1 Rouché's Theorem, Schwarz's Lemma, and Pick's Theorem

1.1 Counting zeros/poles of meromorphic functions and Rouché's theorem

Theorem 1.1. *Suppose f is meromorphic in $\{z : |z - z_0| < R\}$, let $0 < r < R$, and let $\gamma = \{z : |z - z_0| = r\}$. Also, assume that $f(\gamma) \cap \{a, \infty\} = \varnothing$. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - a} \, dz = Z - P,$$

where Z is the number of zeros of $f - a$ in $\{z : |z - z_0| < r\}$ and N is the number of poles of $f - a$ in $\{z : |z - z_0| < r\}$, both counted with multiplicity.

Proof. Let $z_1, z_2, \ldots, z_N$ be the zeros of $f - a$ in $\{z : |z - z_0| < r\}$, and let $p_1, p_2, \ldots, p_M$ be the poles of $f - a$ in $\{z : |z - z_0| < r\}$, with multiplicities. Let

$$g(z) = \frac{f(z) - a}{\prod_{j=1}^N (z - z_j)} \prod_{k=1}^M (z - p_k).$$

Then g is holomorphic, zero free, and pole free in $\{z : |z - z_0| < r\}$. So

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int_\gamma \frac{g'}{g} \, dz &= \frac{1}{2\pi i} \int_\gamma \frac{f'}{f - a} \, dz + \frac{1}{2\pi i} \sum_{j=1}^M \int_\gamma \frac{1}{z - p_j} \, dz - \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_j} \, dz \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'}{f - a} \, dz + M - N. \end{aligned}$$

$\square$

Corollary 1.1 (Rouché). *Let $f, g \in H(\Omega)$, and let $\{z : |z - z_0| < R\} \subseteq \Omega$, where f, g have no zeroes on $\{z : |z - z_0| = R\}$. If $|f - g| < |f| + |g|$ on $\{z : |z - z_0| = R\}$, then on $\{z : |z - z_0| < R\}$, f and g have the same number of zeros, counted with multiplicity.*

Proof. Let $f = f/g$. Then h is meromorphic on $\{z : |z - z_0| < R + \varepsilon\}$ for some $\varepsilon > 0$. Then

$$\frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{h'(z)}{h(z)}\, dz = \#\text{ zeros of } f - \#\text{ zeros of } g.$$

This is also equal to the winding number $n(h(|z - z_0| = R), 0)$, which is zero because $|h - 1| < |h| + 1$, which means that h has winding number 0 around 0. □

1.2 Schwarz's lemma and Pick's theorem

Let $\mathbb{D} = \{z : |z| < 1\}$. Let $\mathbb{T} = \partial\mathbb{D}$. Let $T : \mathbb{D} \to \mathbb{D}$ be a Möbius transformation. such that $T(\alpha) = 0$ with $\alpha \in \mathbb{D}$. Then $T(\partial\mathbb{D}) = \partial\mathbb{D}$, so $T(1/\overline{\alpha}) = \infty$, since T sends symmetric points to symmetric points. So we have

$$T(z) = e^{i\lambda} \frac{z - \alpha}{1 - \overline{\alpha} z},$$

where $\lambda \in \mathbb{R}$.

We have just proved the following.

Theorem 1.2. *Let T be a Möbius transformation. Then $T : \mathbb{D} \to \mathbb{D}$ iff*

$$T(z) = e^{i\lambda} \frac{z - \alpha}{1 - \overline{\alpha} z},$$

where $\alpha \in \mathbb{D}$, and $\lambda \in \mathbb{R}$.

Lemma 1.1 (Schwarz[1])**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$. Then $|f(z)| \leq |z|$, and $|f'(0)| \leq 1$. If equality holds in either case, then $f(z) = e^{i\lambda} z$ for $\lambda \in \mathbb{R}$.*

Proof. We can write $f(z) = a_1 z + z_2 z^2 + \cdots$. Let $g(z) = f(z)/z = a_1 + a_2 z + \cdots$. So $g : \mathbb{D} \to \mathbb{C}$, and g is holomorphic. For $z \leq r < 1$, $|g(z)| \leq 1/r$ by the maximum principle. Then $|g(z)| \leq 1$, so $|f(z)| \leq |z|$. Also, $|g'(0)| \leq 1$, so $f'(0) \leq 1$.

Equality in either case implies $|g(z)| = 1$ for some $z \in \mathbb{D}$. Then $|g| = 1$ on $\mathbb{D}$ by the maximum principle. So $f(z) = e^{i\lambda z}$. □

Corollary 1.2 (Pick's theorem)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic and $z \in \mathbb{D}$. Then*

$$\frac{f'(z)}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2},$$

and equality holds if and only if f is a Möbius transformation.

[1] According to Professor Garnett, entire books have been written on this lemma.

Proof. Take $z_0 \in D$. Let

$$F(z) = \frac{f(\frac{z+z_0}{1+\overline{z}_0 z}) - f(z_0)}{1 - \overline{f(z_0)} f(\frac{z+z_0}{1+\overline{z}_0 z})}.$$

Then $F = S \circ f \circ T$, where $Tz = (z+z_0)/(1+\overline{z}_0 z)$, and $Sw = (w-w_0)/(1-\overline{w}_0 w)$, where $w_0 = f(z_0)$.

Then $F : \mathbb{D} \to \mathbb{D}$, and $F(0) = 0$. So we get $|F'(0)| \leq 1$, and

$$F'(0) = S'(w_0) f'(z_0) T'(0).$$

Observe that

$$T'(0) = 1 - |z_0|^2,$$
$$S'(w_0) = \frac{1}{1 - |w_0|^2}.$$

Then

$$\frac{1}{1-|w_0|^2} |f'(z_0)| (1 - |z_0|^2) \leq 1. \qquad \square$$

1.3 Hyperbolic distance

Definition 1.1. Let γ be a piecewise C^1 arc in $\mathbb{D}$, $\gamma = \{z(t) : a \leq t \leq b\}$. Then the **arc length** of γ is

$$\int_a^b |z'(t)|\, dt$$

We write this as $\int_a^b ds$, where $ds = |z'(t)|\, dt$. This is the length of the parametrized arc $z(a)$ to $z(b)$. This is independent of parametrization by change of variables.

Definition 1.2. Let $z_1, z_2 \in \mathbb{D}$, and let

$$\rho(z_1, z_2) = \inf_{\substack{\gamma(a)=z_1 \\ \gamma(b)=z_2}} \int_\gamma \frac{1}{1-|z|^2}\, ds.$$

This is the **hyperbolic distance** from z_1 to z_2.

This is a metric. Moreover, if T is a Möbius transformation,

$$\rho(Tz_1, Tz_2) = \rho(z_1, z_2).$$